Combinatorial Manifolds:
Bonnet-Myer's Theorem
(RTG)

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1 Introduction and Motivation

During my RTG project, I investigated a paper, Positively Curved Combinatorial 3-Manifolds, by Aaron Trout in which he proves a combinatorial version of the Bonnet Myer's Theorem. Therefore, in this paper we will be discussing the background material necessary to construct and understand positively curved combinatorial manifolds. Then we will cover some of the process of proving this theorem inserting slightly more detailed proofs with more diagrams. Finally, we will observe the reason why we looked at this paper which involves generalizing the result for different edge lengths to make it a bit more analogous to the Riemannian version involving Ricci Curvature which it is based on [2]. The resulting Theorem we will be looking at is the following:

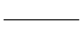
Theorem 1. *[1, Thm. 1.1 on page 2]: A combinatorial* 3*-manifold with edges of degree at most five has edge diameter at most five.*

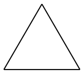
2 Background and Construction of Positively Curved Combinatorial Manifolds

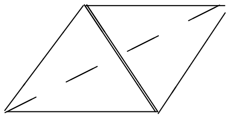
2.1 Simplicies and Simplicial Complex

We begin by looking at definitions of and involving Simplicies and a Simplicial Complex; these are the foundation for a Combinatorial Manifold. A **convex set** is a set in which each pair of points in the set can be connected by a straight line which is entirely contained in the set. The **convex hull** of a set of points is the smallest convex set containing all the points in the set. A **polytope** is a geometric object with flat sides. An n-**simplex** is an n-dimensional polytope which is the convex hull of its $n+1$ vertices. Another way we think of a **simplex** is as a generalization of a triangle into different dimensions:[3, Pages 102-104]

0-simplex = vertex: •

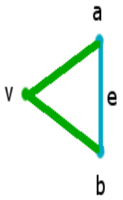
1-simplex = edge: 

2-simplex = triangle: 

3-simplex = tetrahedron: 

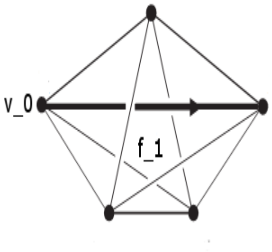
We now will look at a few more definitions which are critical to our construction and our proof process later. A **face** of an n-simplex is an $(n-1)$-simplex which lies on the n-simplex. Some examples are: A face of an edge is a vertex, A face of a triangle is an edge, and A face of a tetrahedron is a triangle. A **simplicial complex** is a set of simplicies such that faces of simplices in the complex are also in the complex and the intersection of any two simplicies is a face of both of them. The **star** of a simplex is the set of other simplicies with faces in the simplex. The **closure of the star** of a simplex is the smallest simplicial complex containing all the simplicies of the star. The **link** of a simplex is the closure of its star minus its star [4]. We **adjoin** two simplices by connecting each vertex of one simplex to each vertex of the other simplex via 1-simplices [3].

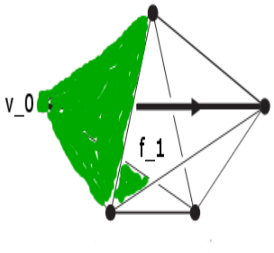
Figure 1: Adjoining the vertex(0-simplex) v to the edge(1-simplex) e by connecting each vertex a and b of e to each vertex of v(v itself) using the green edges.



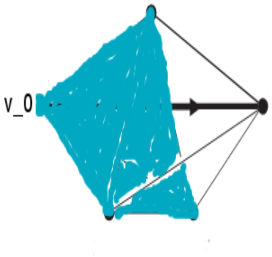
Example of the star, closure of the star, and link of a vertex:

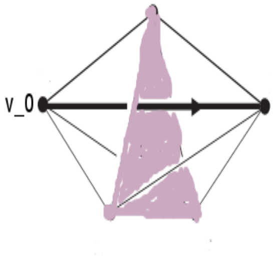
[1, Original diagram from Figure 1 on Page 4]





Star of v_0





Closure of Star of v_0

Link of v_0

2.2 Positively Curved Combinatorial Manifold

We define a **(Boundaryless) Combinatorial n-manifold**, M^n, as a simplicial complex in which the link of each k-simplex is an $(n-k-1)-$sphere [1]. For example, in a 3-manifold, M^3, the link of an edge will be congruent to S^1(circle) and the link of a vertex will be congruent to S^2. We will be dealing with M^n as defined above but with the extra condition that all edges have length 1. In order to understand what positively curved means in this situation, we need the definition of degree of a simplex. The **degree** of a k-simplex is the number of n-simplices in which the k-simplex is part of M^n.

Curvature is the measure of the flatness of a space and positive curvature is a notion of an upward contour. We define a combinatorial manifold to have **Positive curvature** when the total angle around each $(n-2)$-simplex σ is less than 2π. Since our edges all have length 1, that is that the degree of each $(n-2)$-simplex times the angle between the two $(n-1)$-simplices whose intersection is the $(n-2)$-simplex must be less than 2π. For example, in M^3, for positive curvature, the total angle around each edge(1-simplex) σ is $\deg(\sigma)\cos^{-1}(\frac{1}{3}) \approx \deg(\sigma)1.23 < 2\pi \approx 6.28$ and therefore, $\deg(\sigma) < 5.10$. The $\cos^{-1}(\frac{1}{3})$ occurs in this case because it is the dihedral angle of a regular tetrahedron, the angle between any two faces of the tetrahedron. So, for M^3, the degree of an edge can be at most 5 in order to maintain positive curvature. Therefore, we have a definition for a combinatorial manifold being positively curved in terms of degree of each $(n-2)$-simplex.

M^n is **positively curved** if each $(n-2)$-simplex has degree at most

$$\epsilon(n) = \begin{cases} 5, & n = 2, 3 \\ 4, & n \geq 4 \end{cases}$$

2.3 How do we travel around our combinatorial manifolds?

Now that we have constructed our combinatorial manifolds, we need to figure out how to travel around them from vertex to vertex. We can naturally move along the edges, but we want to introduce two other forms of transportation which we call hops and jumps. Recall from above that we **adjoin** two simplices by connecting each vertex of one simplex to each vertex of the other simplex via 1-simplices. When we take two distinct vertices(0-simplex) and adjoin each of them to an $(n-1)$-simplex, there exists an **n-Hop** between the two vertices which travels through the barycenter(center of mass) of the $(n-1)$-simplex [1]. We say that the $(n-1)$-simplex is transverse to the hop.

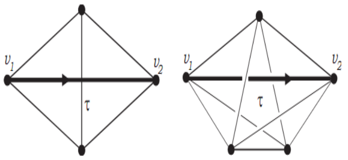
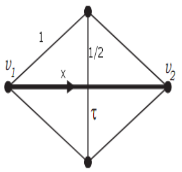


Figure 1: Two and three dimensional hops

[1, Figure 1 page 4]

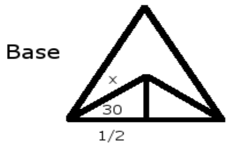
The length of an n-Hop is $H_n = \sqrt{2 + \frac{2}{n}}$.

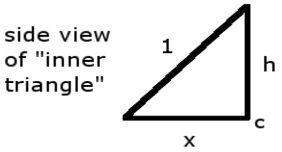
We are concerned with distance between vertices, so the length of the n-hop is very important. We are most concerned with the 2 and 3 dimensional cases, so let's calculate their lengths.



[1, Original diagram from Figure 1 page 4]

To calculate the 2-hop length, we observe that the hop intersects the edge at its center creating four right triangles each with hypotenuse of length 1 since our edges have lengths 1. Therefore, half of the length of the 2-hop is one of the other sides of one of these right triangles which has length $x = \sqrt{1^2 - (\frac{1}{2})^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$ by the Pythagorean Theorem. Thus, the total length of the 2-hop is $2x = 2\frac{\sqrt{3}}{2} = \sqrt{3}$.





To calculate the 3-hop length, we similarly observe that the hop intersects the 2-simplex at its barycenter and that each half of the hop is the height of a tetrahedron from this center of its base. The barycenter of the 2-simplex is the point where the three medians of the triangle intersect. Since these medians bisect both the edge and the vertex they touch, we get a right triangle as indicated above with angle $30\,\deg = \dfrac{\pi}{6}$ and hypotenuse we call x. Then $\cos(\dfrac{\pi}{6}) = \dfrac{\frac{1}{2}}{x}$ and therefore, $x = \dfrac{\frac{1}{2}}{\cos\frac{\pi}{6}} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$. Now, using the diagram to the right of a side view of an inner triangle we are inserting to help with calculations with c the barycenter of the base, we see that $h = \sqrt{1^2 - x^2} = \sqrt{1 - \frac{3}{9}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}}$ is the height of a regular tetrahedron. Thus, the length of a 3-hop is $2\sqrt{\frac{2}{3}} = \sqrt{\frac{4(2)}{3}} = \sqrt{\frac{8}{3}} = \sqrt{2 + \frac{2}{3}}$.

A **jump** is formed by taking two distinct vertices(0-simplex) and adjoining each of them to a distinct 1-simplex while also adjoining the two distinct 1-simplices to each other [1]. This only exists in three dimensions and higher since adjoining the 1-simplices forms a tetrahedron. We say that each of the edges(1-simplices) is transverse to the jump.

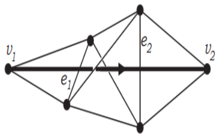


Figure 2: A jump

[1, Figure 2 page 4]

The length of a jump is $J = \sqrt{3} + \frac{1}{2}\sqrt{2}$.

Our paths between vertices by which we travel around the combinatorial manifolds will consist of combinations of vertices, edges, hops, and jumps. The **edge distance** between vertices v, w is denoted by $d_1(v, w)$ and is the minimum length via edges between the two vertices. Note that since the edges are all of the equal length 1, this is also the minimum number of edges connecting the two vertices. For a general metric this will still be the minimum length via edges but will not typically be just the number of edges since the edges will possibly be of different lengths and not length 1. The **distance** between vertices v, w is denoted by $d(v, w)$ and is the minimum length between the two vertices using combinations of edges, hops, and jumps. Note that this is the length of the minimum path between the two vertices. These distance measurements are very important because our goal theorem involves edge-diameter in its result and edge-diameter is the minimum edge distance connecting any two vertices in the manifold.

We will need to minimize paths. Therefore, we define a minimal path and an almost minimal path. A **minimal path** is a path with length equal to that of the distance between the endpoints of the path. An **almost minimal path** is a path in which every proper subpath is minimal[1].

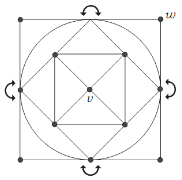
3 Bonnet Meyer's Theorem

Theorem 2. *[1, Thm. 1.1 on page 2]: A combinatorial* 3*-manifold with edges of degree at most five has edge diameter at most five.*

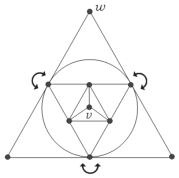
3.1 2-dimensional case

Since positively curved surfaces with triangular faces and equal edge lengths are regular polytopes, we can classify all positively curved combinatorial 2-manifolds as either the Octahedron, the Tetrahedron, and the Icosahedron. Below are flattened pictures of the three with a slight refinement in the center. Note that refining with more vertices does not affect the conditions or results we look at.

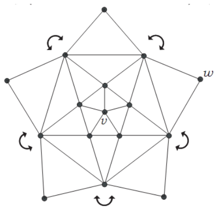
[1, Page 6]



Octahedron



Tetrahedron



Icosahedron

The following results for the 2-dimensional manifolds can be found by inspection using these diagrams.

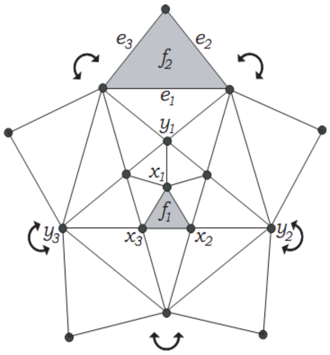
Lemma 1. *[1, Lemma 3.1]: If P is a minimal path with one internal vertex x then $deg(x) = 5$ and P has length $1 + H_2$. Moreover, given the initial hop or edge in P the remainder is uniquely determined.*

Corollary 1. *[1, Corollary 3.2]: $d(v, w) \in \{0, 1, H_2, 1 + H_2\}$ for any vertices $v, w \in M^2$. $d(v, w) = 1 + H_2$ if and only if $d_1(v, w) = 3$.*

Corollary 2. *[1, Corollary 3.3]: For a fixed vertex v in M^2, we have $d_1(v, w) = 3$ for at most one vertex w.*

Corollary 3. *[1, Corollary 3.4]: Suppose* $d_1(v, w) = 3$ *for vertices* v, w *in a positively curved surface* M^2*. Then, we have:*

1. *deg*$(v) =$ *deg*(w)
2. *Any minimal hop beginning on* v *ends on a vertex in* $Lk(w)$*(Link of* w*).*
3. *Any vertex in* $Lk(v)$ *is the beginning of a minimal hop to* w*.*



[1, Page 7]

The following results for the 2-dimensional manifolds can be found by inspection of the above diagram.

Lemma 2. *[1, Lemma 3.5]: For all* 2*-simplicies* f_1, f_2 *and* 1*-simplicies* e_1, e_2 *in a positively curved surface* M^2 *we have:*

1. $d(f_1, f_2) \leq H_2$
2. $d(St(e_1), e_2) \leq H_2$ *(*$St(e_1)$ *is the Star of* e_1*)*

Moreover, for fixed f_1*(or* e_1*) at most one* f_2*(or* e_2*) gives equality, and this occurs only when* M^2 *is an icosahedron.*

Lemma 3. *[1, Lemma 3.6]: Let* M^2 *be an icosahedron.*

1. *Suppose* $d(f_1, f_2) = H_2$ *for* 2*-simplices* $f_1, f_2 \in M^2$*. For each vertex* x_i *in* f_1 *there is a unique edge* e_i *in* f_2 *and vertex* $y_i \in Lk(e_i)$ *so that* $[x_i, y_i]$ *is an edge. Similarly, each* e_i *in* f_2 *gives unique* $y_i \in Lk(e_i)$ *and* x_i *in* f_i *such that* $[x_i, y_i]$ *is an edge.*
2. *Suppose* $d(St(e_1), e_2)) = H_2$ *for edges* $e_1, e_2 \in M^2$*. An edge connects each vertex in* $Lk(e_1)$ *to exactly one vertex in* $Lk(e_2)$*(and vice-versa).*

3.2 3-Dimensional Case

Using the links of vertices in M^3, we reduce to an M^2 case since our combinatorial manifolds are defined such that the link of a vertex is a 2-sphere. We used the following lemma quite a bit:

Lemma 4. *[1, Lemma 2.2]: Suppose* $v \in M^3$ *is a vertex. Vertices* $w_1, w_2 \in Lk(v)$ *are connected by an* 3*-hop within* $\overline{St(v)}$ *iff they are connected by an* 2*-hop within* $Lk(v)$*.*

Let M^3 be a positively curved combinatorial 3-manifold.

Lemma 5. *[1, Lemma 4.1]: Suppose* P *is a minimal path which passes through the vertices* v_0, v_1, v_2 *in order. Let* $L = Lk(v_1)$*(recall: this is the link of* v_1*) and let* $d^L(v, w)$ *be the distance between* v *and* w *within* $Lk(v_1)$*.* $Lk(v_1)$ *is a positively curved combinatorial* 2*-manifold.*

Within $L = Lk(v_1)$*:*

1. *1. If* P *is a two edge path then* $d^L(v_0, v_2) = 1 + H_2$*.*
2. *2. If* P *is a two hop path then* $d^L(f_1, f_2) = H_2$ *where the* 2*-simplices* $f_1, f_2 \in Lk(v_1)$ *are transverse to the hops.*
3. *3. If* P *is a two jump path then* $d^L(St^L(e_1), e_2) = H_2$ *where the edges* $e_1, e_2 \in Lk(v_1)$ *are transverse to the jumps.*

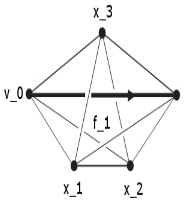
In each case, given $v_0, f_1,$ *or* e_1 *the corresponding* $v_2, f_2,$ *or* e_2 *is uniquely determined. In the last two cases,* $Lk(v_1)$ *is an icosahedron.*

Proof. We will prove each case individually and in the same order as in the lemma.

1. Suppose P is a minimal two edge path between v_0 and v_2 within the link of v_1,$L = Lk(v_1)$. Suppose $d^L(v_0, v_2) < 1 + H_2$. Then since $L = Lk(v_1)$ is a positively curved combinatorial 2-manifold, $d^L(v_0, v_2) = \{0, 1, H_2\}$. Now, if $d^L(v_0, v_2) = 0$ in L, it certainly is 0 in M^3. Similarly, edges don't change since the link is just a portion of our simplicial complex; therefore, if $d^L(v_0, v_2) = 1$ in L, it also does in M^3. Using the Lemma mentioned above, $d^L(v_0, v_2) = H_2$ iff $d(v_0, v_2) = H_3$ in M^3 since the closure of

the star of v_1 is in M^3 but not in L and an $3 - hop$ in the star's closure occurs iff a 2-hop occurs in the link. Therefore, $d(v_0, v_2) \leq H_3$ in M^3 but $d(v_0, v_2) = 2$ since we have a minimal two edge path between v_0 and v_2. We get a contradiction here because $H_3 = \sqrt{2 + \frac{2}{3}} = \approx 1.63 < 2$ but our two edge path was minimal so no shorter path can exist. Thus, $d^L(v_0, v_2) = 1 + H_2$.

2. Suppose P is a minimal two hop path (where the 2-simplices $f_1, f_2 \in Lk(v_1)$ are transverse to the hops) between v_0 and v_2 within the link of v_1,$L = Lk(v_1)$. Then since $L = Lk(v_1)$ is a positively curved combinatorial 2-manifold, $d^L(f_1, f_2) \leq H_2$ ($d^L(f_1, f_2) \neq 1 + H_2$ by Lemma 3. Suppose $d^L(f_1, f_2) < H_2$. Note that this is the distance between two 2-simplices and it is measured as the minimal distance between any vertex in f_1 and any vertex in f_2. Then since $L = Lk(v_1)$ is a positively curved combinatorial 2-manifold, $d^L(f_1, f_2) \leq H_2$ using a result from the two dimensional case. Therefore, the two dimensional case also informs us that if $d^L(f_1, f_2) \leq H_2$, then $d^L(f_1, f_2) = \{0, 1, H_2\}$. Suppose $d^L(f_1, f_2) \leq 1$. Then for any x_i which is a vertex in f_1, $d(v_0, x_i) \leq 1$ simply by the structure of the 3-hop which is transverse to f_1. This can be seen in the diagram below:



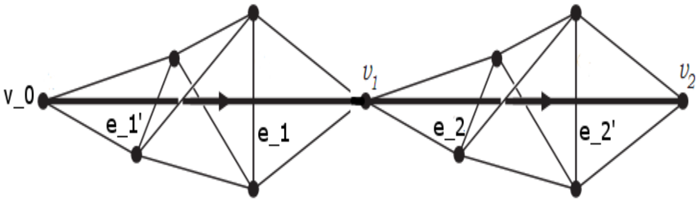
[1, Original diagram from page 4]

Similarly, $d(y_j, v_2) \leq 1$ for any vertex y_j in f_2 by the structure of the 3-hop which is transverse to f_2. Recall that we assumed that $d^L(f_1, f_2) \leq 1$ which means $d^L(x_i, y_j) \leq 1, \forall i, j$ Thus, $d(v_0, v_2) = d(v_0, x_i) + d(x_i, y_j) + d(y_j, v_2) \leq 1 + 1 + 1 = 3 < 2H_3 \approx 3.27$. Although, the two 3-hop path was minimal between v_0 and v_2. Therefore, this contradicts that path being minimal because the path through some x_i, y_j is shorter. Thus, $d^L(f_1, f_2) \leq 1$ is not possible which leaves us with $d^L(f_1, f_2) = H_2$.

3. Suppose P is a minimal two jump path (where the 2-simplices $f_1, f_2 \in Lk(v_1)$ are transverse to the hops) between v_0 and v_2 within the link of v_1,$L = Lk(v_1)$. Then

since $L = Lk(v_1)$ is a positively curved combinatorial 2-manifold, $d^L(St^L(e_1), e_2) \leq H_2$ ($d^L(St^L(e_1), e_2) \neq 1 + H_2$) by Lemma 3. Let e_2' be the other transverse edge in the jump transverse to e_2 and e_1' be the other transverse edge in the jump transverse to e_1 as seen below. Although, note that this is just a portion of the simplicial complex and these 2-simplices which appear to be intersecting at a vertex actually live inside tetrahedra because remember 2-simplices intersect at edges. This diagram is simply a local representation of the jumps themselves.

[1, Original diagram from page 4]



The next steps of this proof are a bit hard to visualize from these flat diagrams and I actually built a rough model of tetrahedra out of paper and tape in order to precisely deal with this type of step. If one does this then we can see that using the structure of the jump and the positively curved assumption of $deg(e_1) \leq 5$, we can conclude that $d(v_0, x) \leq 2$ for each vertex $x \in Lk^L(e_1)$ where $Lk^L(e_1) = Lk(e_1) \cap L$. We can similarly conclude that $d(y, v_2) \leq H_3$ for each vertex y in e_2 since $deg(e_2') \leq 5$. Therefore, $d(v_0, v_2) = d(v_0, x) + d(x, y) + d(y, v_2) \leq 2 + 1 + H_3 \approx 4.53 < 2J \approx 4.88$. This contradicts the two jump path being minimal and thus $d^L(St^L(e_1), e_2) \leq H_2$ is not true and thus $d^L(St^L(e_1), e_2) = H_2$.

□

Lemma 6. *[1, Lemma 4.2]: A minimal path contains either all edges, all hops, or all jumps.*

Corollary 4. *[1, Corollary 4.3]: If two non-trivial minimal paths of equal length pass through the same first two simplices then the paths are identical.*

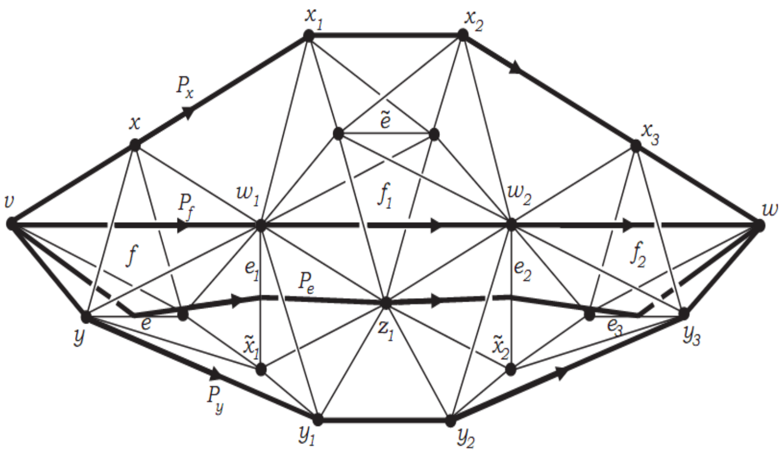
Lemma 7. *[1, Lemma 4.4]: A minimal path contains at most two jumps.*

Lemma 8. *[1, Lemma 4.5]: Suppose P_x is a five edge almost minimal path from v to w with first internal vertex $x \in Lk(v)$. Then, each 2-simplex $f \in Lk(v)$ with x in f is transverse to the first hop in a three hop path P_f from v to w.*

I will also highlight the proof of the following lemma because it shows the more of how the links of vertices help by reducing to the 2-dimensional case.

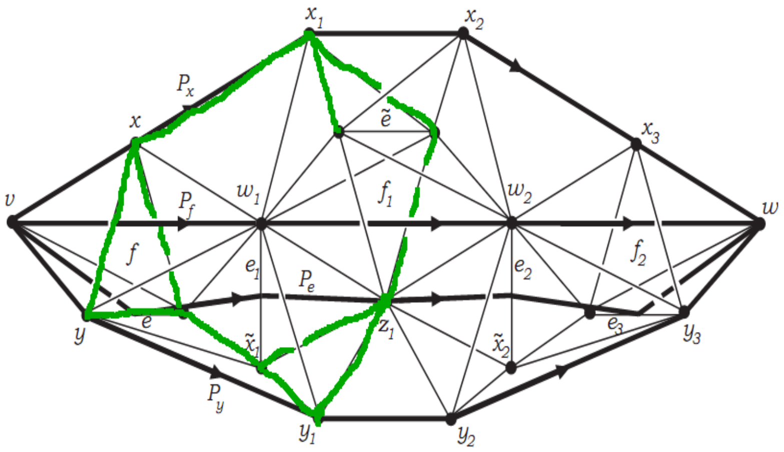
Lemma 9. *[1, Lemma 4.7]: Suppose P_f is a three hop almost minimal path from v to w with the 2-simplex $f \in Lk(v)$ transverse to the first hop. Then, each edge e in f is transverse to the first jump in a two jump path P_e from v to w.*

Proof. Suppose P_f is a three hop almost minimal path from v to w with the 2-simplex $f \in Lk(v)$ transverse to the first hop such that this path travels through the vertices v, w_1, w_2, w in that order and where f_1 and f_2 are transverse to the second and third hops, respectively. A possible version of this can be seen in the diagram below:



[1, Page 11]

[1, Original diagram from page 11]



As an example, the edges and vertices which are in $Lk(w_1)$ have been colored green.

Using Lemma 5, since we have a three hop path, the w_1 and w_2 are the equivalent of v_1 in Lemma 5 and so within the $Lk(w_i)$ we have $d(f, f_1) = H_2 = d(f_1, f_2)$. Now, we will continue to work in $Lk(w_i)$ with it as our M^2. Then, Lemma 3 tells us that for each e in f, there is a unique vertex $\tilde{x_1} \in Lk(e)$ and a unique vertex z_1 in f_1 such that there is an edge between $\tilde{x_1}$ and z_1. If we adjoin the initial vertex v to the edge e, adjoin e to the edge between $\tilde{x_1}$ and w_1, and adjoin the edge between $\tilde{x_1}$ and w_1 to z_1 and we can construct a jump from v to z_1. Then we repeat the process with z_1 as the beginning of a subpath and using $Lk(w_2)$. This ultimately gives us a jump from z_1 to w by an almost identical process. This results in a two jump path from v to w with e transverse to the first jump.

$\square$

Corollary 5. *[1, Corollary 4.9]:* $d(v,w) \in \{0,1,H_3,2,J,3,2H_3,4,2J\}$ *for any vertices* $v,w \in M^3$

Corollary 6. *[1, Corollary 4.10]:* $d(v,w)=2J \Rightarrow d_1(v,w)=5:$ *Suppose* P_e *is a two jump minimal path from* v *to* w *and the edge* $e \in Lk(v)$ *is transverse to the first jump. Then, each vertex* y *in* e *is the first internal vertex of a five edge path* P_y *from* v *to* w.

These give us the result of the theorem we desired. This is because we assumed positive curvature and thus that the degree of edges would be at most five and the edge diameter is the minimum distance between any two vertices of the manifold via edges only; so, since we know the minimum distance between any two vertices v,w is $d(v,w) \in \{0,1,H_3,2,J,3,2H_3,4,2J\}$ and $2H_3 < 2J, 4 < 5$ then since $d(v,w)=2J \Rightarrow d_1(v,w)=5$, $d_1(v,w) \leq 5$ for any two vertices $v,w \in M^3$.

Theorem 3. ***[1, Thm. 1.1 on page 2]:A combinatorial 3-manifold with edges of degree at most five has edge diameter at most five.***

4 Conclusion:

The next step is generalize edge lengths to see if the result of the theorem will still hold but with a modification to the hypotheses. Since the edge lengths will vary by what we hope forms a metric, the requirement of positive curvature will need to change slightly because it is based on edge lengths being equal. The path types we use to travel around our combinatorial manifolds included combinations of edges, hops, and jumps. This will not change when we change the lengths of edges since the hops and jumps exist based on adjoining simplicies which doesn't depend on lengths. Although, the lengths of the edges, hops, and jumps will change but should be proportionate. Due to this change, the minimal path lengths might change.

Since curvature for 2-dimensional manifolds is based on the total angle around a vertex, the edge lengths changing won't change curvature in 2-dimensional manifolds. Therefore, the 2-dimensional manifolds will continue to be either the octahedron, tetrahedron, or icosahedron. This is good because the 2-dimensional results depend heavily on this and are imperative to proving the 3-dimensional case. When the edge lengths vary, we expect a lower bound on curvature will be necessary rather than just positive curvature in order for the theorem to continue to hold true for the 3-dimensional case; if this occurs, the theorem will be more analogous to the Riemannian version which deals with Ricci curvature [2].

References

[1] Aaron Trout, *Positively Curved Combinatorial* 3*-Manifolds*, Chatham University, 2010.

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[3] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[4] Munkres, J. R. *Elements of Algebraic Topology.* New York: Perseus Books Pub., 1993. Used by way of Wolfram Mathworld.